Pergamon

0021–8928(95)00076-3

J. Appl. Maths Mechs, Vol. 59, No. 4, pp. 657–659, 1995

0021–8928/95 $24.00+0.00

THE EVOLUTION OF AN ALFVÉN DISCONTINUITY IN MAGNETOHYDRODYNAMICS†

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(*Received* 17 *May* 1993)

The evolution of an Alfvén discontinuity in magnetohydrodynamics is investigated taking dissipative processes into account together with an arbitrary value for the angle of rotation of the transverse component of the magnetic field.

The existence of a stationary structure in the form of a travelling wave for plane Alfvén discontinuities has been demonstrated in [1]. In this paper we use a model equation for weakly-linear Alfvén waves to obtain a self-similar solution describing the evolution of an arbitrary discontinuity.

1. We start by considering a system of one-dimensional magnetohydrodynamic equations with fluid and magnetic dissipation. It is assumed that all quantities depend only on the variables x and t. This system has the dimensionless form

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial s}\frac{\partial s}{\partial x}-\frac{1}{2}\frac{\partial B^2}{\partial x}+\frac{1}{R_e}\frac{\partial^2 u}{\partial x^2}$$

$$\rho\left(\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}\right)=B_x\frac{\partial B_y}{\partial x}+\frac{1}{R_e}\frac{\partial^2 \upsilon}{\partial x^2},\quad \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}\right)=B_x\frac{\partial B_z}{\partial x}+\frac{1}{R_e}\frac{\partial^2 w}{\partial x^2}$$

$$\rho T\left(\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}\right)=\frac{1}{R_e}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{R_m}\left\{\left(\frac{\partial B_z}{\partial x}\right)^2+\left(\frac{\partial B_y}{\partial x}\right)^2\right\}$$

$$\frac{\partial B_y}{\partial t}=-\frac{\partial}{\partial x}\left(uB_y-\upsilon B_x\right)+\frac{1}{R_m}\frac{\partial^2 B_y}{\partial x^2},\quad \frac{\partial B_z}{\partial t}=\frac{\partial}{\partial x}(wB_x-uB_z)+\frac{1}{R_m}\frac{\partial^2 B_z}{\partial x^2}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0,\quad B_x=\text{const}$$

Here u, υ, w are the components of the velocity vector, B_x, B_y, B_z are the components of the magnetic induction vector, R_e is the Reynolds number, and R_m is the magnetic Reynolds number. The system is reduced to dimensionless form using the Alfvén velocity and characteristic values for the density, magnetic field, entropy and temperature.

We represent B_y and B_z in the form $B_y = B\sin\theta$, $B_z = B\cos\theta$, where B is the magnitude of the transverse component of the magnetic field and θ is the angle of the direction of the magnetic field in the (y, z) plane.

Below we make the important assumption that the dissipation is small, and is represented in the form

$$\varepsilon = 1/R_e + 1/R_m \tag{1.1}$$

where ε is a small parameter.

We change the independent variables using the formulae

$$\xi=\varepsilon(x-t\cos\alpha),\quad \tau=\varepsilon^3 t \tag{1.2}$$

(where α is the angle between the x axis and the unperturbed magnetic field, and $\cos\alpha$ is the dimensionless Alfvén velocity).

Then, using the methods of [2], expanding all variables in powers of ε and substituting into the original system, we obtain, as in [1], the model equation

†*Prikl. Mat. Mekh.* Vol. 59, No. 4, pp. 685–687, 1995.

$$\frac{\partial\theta}{\partial\tau}-\frac{1}{2}\left\{\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial\theta}{\partial\xi}\int\limits_{-\infty}^{\xi}\left(\frac{\partial\theta}{\partial\xi}\right)^2 d\xi\right\}=0 \tag{1.3}$$

In Eq. (1.3) θ is the lower-order term of the expansion in powers of ε of the angle of rotation θ. Lower terms in the expansion of the components of $\mathbf{v}$ and $\mathbf{B}$ are expressed in terms of θ as follows:

$$B_{y0}=\sin\alpha\sin\theta,\quad B_{z0}=\sin\alpha\cos\theta,\quad \upsilon_0=-\sin\alpha\sin\theta,\quad w_0=\sin\alpha(1-\cos\theta)$$

Note that according to the expansion procedure the first non-zero terms in the expansions of the other quantities (ρ, u, s, T, p) have higher orders. It is also significant that if thermal conductivity and second viscosity are included amongst the dissipation mechanisms, the form of Eq. (1.3) is unchanged.

2. We shall seek a self-similar solution of (1.3) in the form

$$\theta=\theta(y),\ y=2\xi\tau^{-1/2} \tag{2.1}$$

Substituting (2.1) into (1.3) we obtain the equation

$$y\theta'+4\theta''+4\theta'\int\limits_{-\infty}^{y}\theta'^2 dy=0 \tag{2.2}$$

Here and below the prime denotes differentiation with respect to y. Using the replacement

$$\theta'=\lambda,\ \theta''=\mu(\lambda) \tag{2.3}$$

we obtain a first-order equation with a solution of the form

$$\mu^2=-\tfrac{1}{2}\lambda^2\ln|\lambda|-\lambda^4-\lambda^2 c,\quad c=-a^2-\tfrac{1}{2}\ln a,\quad a>0 \tag{2.4}$$

In accordance with (2.3) we obtain a solution in parametric form with parameter λ

$$d\theta=\lambda\mu^{-1}d\lambda,\quad dy=\mu^{-1}d\lambda \tag{2.5}$$

Substituting the first of the equations obtained into (2.4), we obtain the relation

$$\theta(\lambda)=\pm\int\limits_{\lambda_0}^{\lambda}f(\lambda)d\lambda,\quad \text{where}\quad f(\lambda,a)=\left(\tfrac{1}{2}\ln\left|\frac{a}{\lambda}\right|+a^2-\lambda^2\right)^{-1/2}, \tag{2.6}$$

which is shown in Fig. 1. The point (0, 0) in Fig. 1 corresponds to the state as $y\to-\infty$; the points $(0, 0_*)$ and $(0, -\theta_*)$ correspond to two branches of the solution as $y\to-\infty$. We now restrict our considerations to the case when $\theta>0$, since the two branches are symmetric. When λ varies from 0 to a the plus sign is chosen in formula (2.6); at the point $(a, \theta_*/2)$ the sign changes.

The dependence of θ_* on a is expressed by the formula

$$\theta_*(a)=2\int\limits_0^a f(\lambda,a)d\lambda \tag{2.7}$$

It follows from (2.7) that

$$\lim_{a\to0}\theta_*(a)=0,\quad \lim_{a\to\infty}\theta_*(a)=\pi,\quad \frac{d\theta_*}{da}>0$$

Hence, the solution obtained gives the evolution of the Alfvén discontinuity with an arbitrary angle of rotation of the transverse component of the magnetic field θ_*, $-\pi<\theta_*<\pi$.

The dependence of θ on y given by (2.7) is shown in Fig. 2. The curve $\theta(y)$ possesses central symmetry about the point with coordinates $(\sigma(a), \theta_*/2)$ (the centre of discontinuity), where $\sigma(a)$ is given by the relation

$$\sigma(a)=-4\int\limits_0^a f(\lambda,a)\lambda d\lambda \tag{2.8}$$

The tangent of the angle of inclination of the tangent to the curve $\theta(y)$ at the point $\sigma(a)$ is equal to a.

We define the width of the discontinuity δ to be the difference between the abscissae of the points of intersection of the tangent to the curve $\theta(y)$ at the point $(\sigma, \theta_*/2)$ with the lines $\theta=0$ and $\theta=\theta_*$. From Fig. 2 we have $\delta(a)=\theta_*(a)/a$, with $\delta(a)$ diminishing monotonically as a increases. In the original dimensionless variables the width of

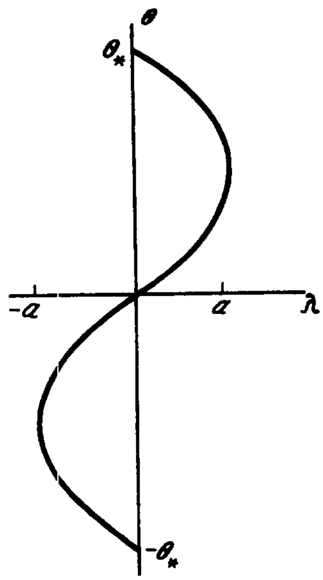

Fig. 1.

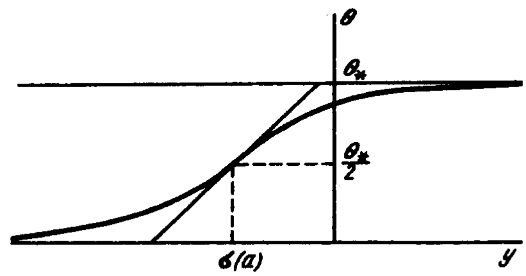

Fig. 2.

the discontinuity Δ can be expressed using (2.1), (1.1) and (1.2) in the form

$$\Delta = \theta_*(a)a^{-1}(1/R_e + 1/R_m)^{1/2}t^{1/2} \tag{2.9}$$

Formula (2.9) agrees with the results of the solution of the problem of the velocity of spread of the Alfvén discontinuity zone in the linear formulation [3].

The rate of widening of the discontinuity zone decreases as θ_* increases. In the limit as $\theta_* \to \pi$ the rate of spread tends to zero (which is consistent with the fact that when $\theta_* = \pi$ the Alfvén discontinuity has a stationary structure). The centre of the discontinuity moves along the gas with a time-dependent velocity

$$V_* = \cos\alpha - \tfrac{1}{4}\sigma(a)t^{-1/2}(1/R_e + 1/R_m)^{1/2}$$

where the function $\sigma(a)$ is given by formula (2.8).

Note that these things do not happen in the linear formulation.

We wish to thank N. Ye. Sysoyev for useful discussions.

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Translated by R.L.Z.